Maths Mastery

Developing Mathematical understanding:

The C-P-A approach

Today’s Aims:

- Today we will discover how the concrete, pictorial, abstract (CPA) approach helps pupils to develop a deep understanding of maths as part of mastery learning
- Explore ideas and reflect on how you can use the CPA approach to support your child in maths

Why CPA?

Children (and adults!) can find maths difficult because it is abstract.

The CPA approach builds on children's existing knowledge by introducing abstract concepts in a concrete and tangible way. It involves moving from concrete materials, to pictorial representations, to abstract symbols and problems.

The CPA framework is so established in Singapore maths teaching that the Ministry of Education will not approve any teaching materials that do not use the approach.

CPA is at the heart of MATHS MASTERY.

Let's break it down!

<u>Concrete</u> Pictorial Abstract (CPA) is a three step instructional approach that has been found to be highly effective in teaching math concepts.

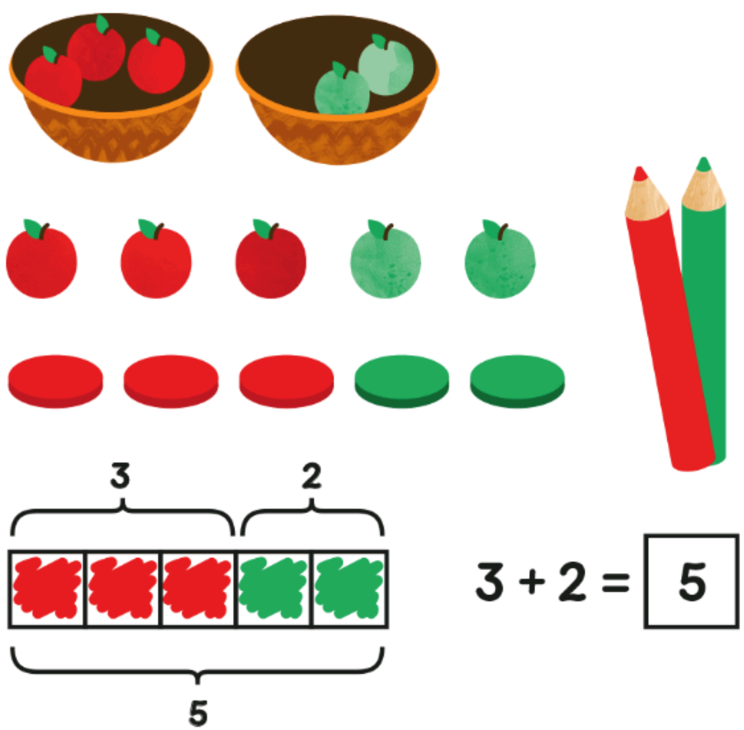
Concrete = The 'doing' stage – physically moving objects to explore a concept. This helps bring the maths to life. Every abstract concept is first introduced using physical, concrete objects.

<u>For example</u>, if a problem involves adding pieces of fruit, children can first handle actual fruit. From there, they can progress to handling abstract counters or cubes which represent the fruit.

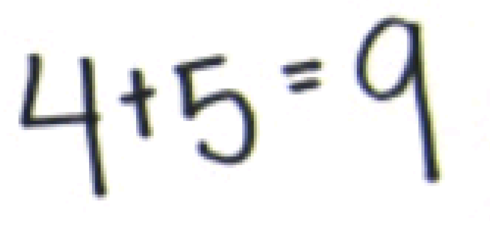
Pictorial = The ‘seeing’ stage – images used to represent the objects.

This stage encourages children to make a mental connection between the physical object they just handled and the abstract pictures, diagrams or models that represent the objects from the problem.

Building or drawing a model makes it easier for children to grasp difficult abstract concepts (for example, fractions). Simply put, it helps children visualise abstract problems and make them more accessible.

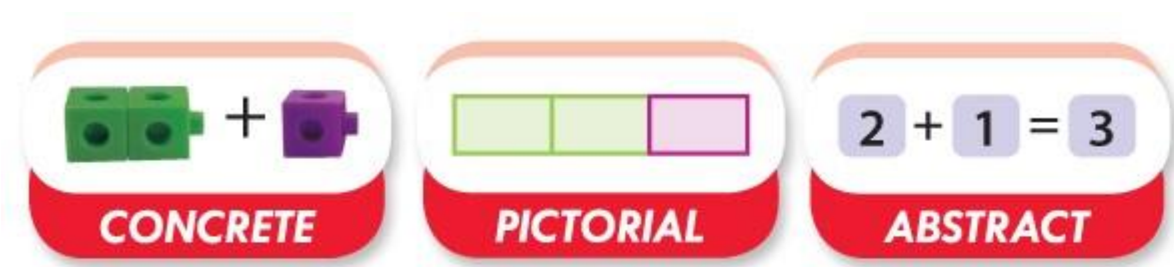


Abstract = The ‘abstract’ stage – symbols and numbers are used to model the problem or calculation. The teacher uses operation symbols (+, –, x, /) to indicate addition, multiplication, or division.



MAKING CPA WORK

Although we’ve presented CPA as three distinct stages, a skilled teacher will go back and forth between each stage to reinforce concepts.



Nothing new here!

Giving children objects and drawings to help them to understand key concepts isn't anything new.

So, what is it that makes this approach so valuable to the study of maths and particularly to the teaching for mastery?

Firstly, CPA is not about getting the answer quickly.

- Concrete manipulatives are used to help children work through new concepts and challenging questions and provide a transition to pictorial and abstract.

After all, maths lessons aren't about teaching tricks; they are about giving children the tools to understand the problem in front of them.

- Interestingly, in a mastery classroom, there <u>doesn't have to be a linear progression</u> from concrete to pictorial to abstract.

Instead, teachers apply a cyclical approach.

<u>*For example*</u>, even when a pupil has worked out the answer using an abstract method, it is worth asking them to use concrete manipulatives to convince others that they are correct.

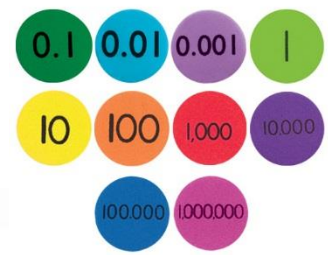
Secondly, CPA is for everyone!

- Useful for all abilities and ages. Concrete manipulatives are a common feature of KS1 classrooms across the country. By KS2, they used to barely exist and were only occasionally brought out for children who were struggling.
- Mastery teaching encourages the use of concrete manipulatives in any lesson as there is value in KS2 children having a variety of equipment to aid their thinking. For these children, concrete objects can often kick-start learning about a new concept and are gradually abandoned as they progress through the lesson.

Finally, CPA is a way to deepen and clarify mathematical thinking.

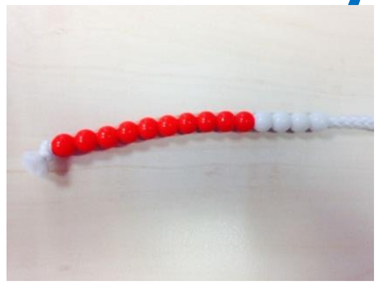
- Children are given the opportunity to discover new ideas and spot the patterns, which will help them reach the answer. From the start of Reception, we introduce CPA as three interchangeable approaches, with pictorial acting as the bridge between concrete and abstract.
- When teaching for mastery, the CPA approach helps learners to be more secure in their understanding, as they have to prove that they have fully grasped an idea. Ultimately, it gives children a **firm foundation for future learning.**

Concrete representation



Concrete representation

Activity 2a



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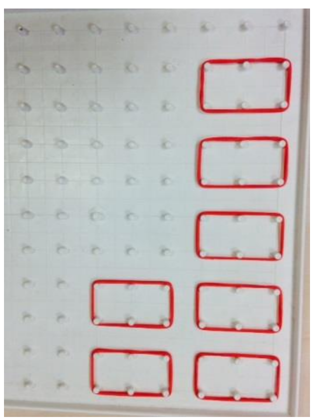
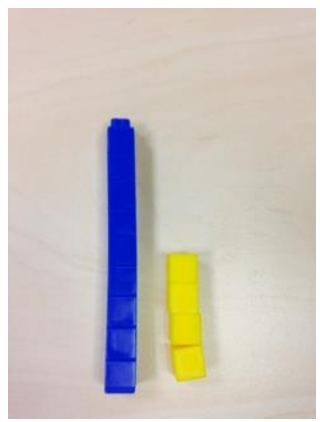
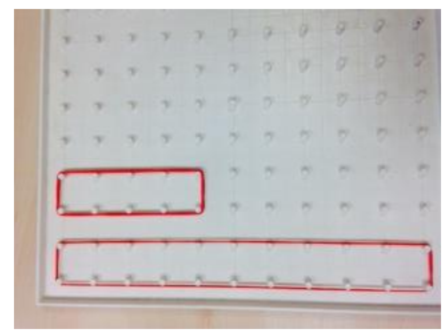
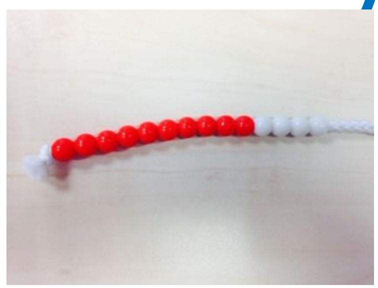






Activity 2b



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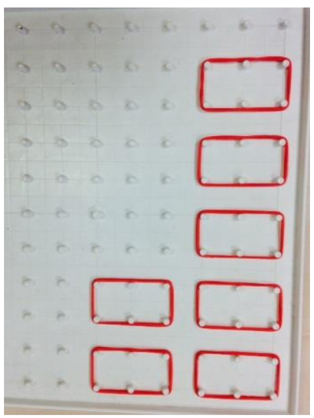
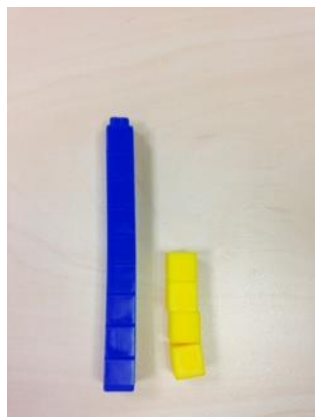
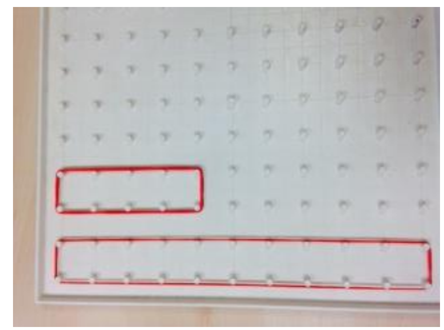




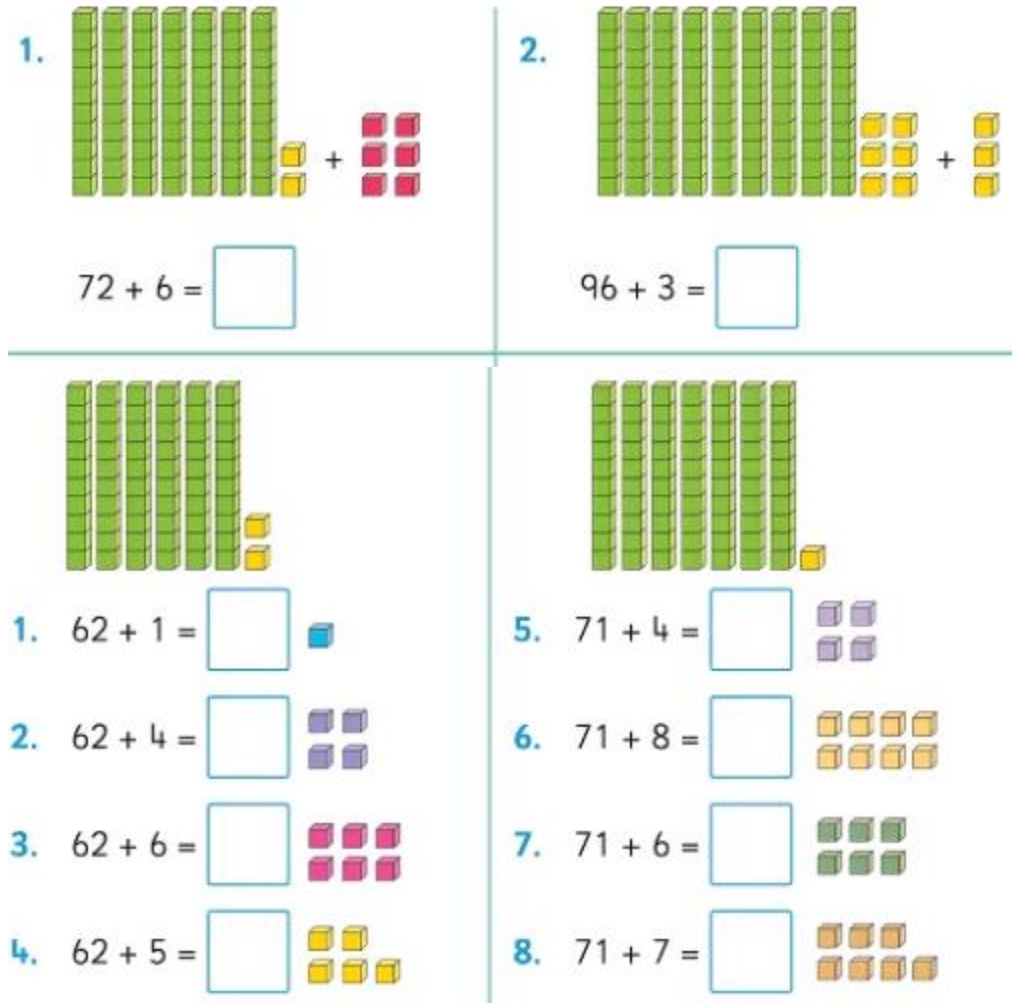


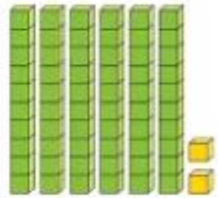
What does CPA look like in different areas of maths?

Addition without regrouping/exchanging

How might this look using PV counters instead?
What about beadtsrings?

Which manipulatives are best?



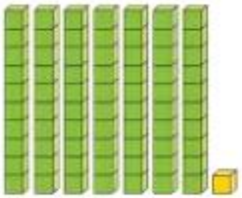


1. 62 + 1 = ☐

2. 62 + 4 = ☐

3. 62 + 6 = ☐

4. 62 + 5 = ☐



5. 71 + 4 = ☐

6. 71 + 8 = ☐

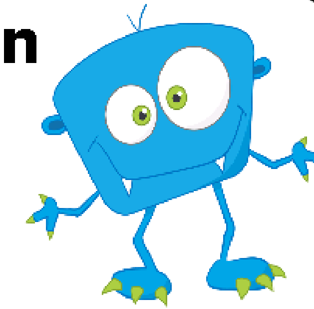
7. 71 + 6 = ☐

8. 71 + 7 = ☐

1.	2.	3.	4.	5.
T U	T U	T U	T U	T U
5 4	8 1	9 2	6 4	7 4
+ 5	+ 7	+ 6	+ 4	+ 1

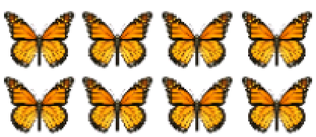
When children are secure, a formal method is taught.

Let's look deeper at why multiplication can be worked out in this way.



Multiplication is **COMMUTATIVE**.

Two or more numbers can be multiplied in <u>any order</u> (swapped around) and the same answer will be given.



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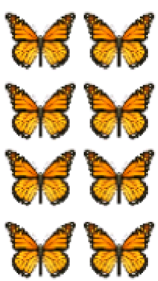
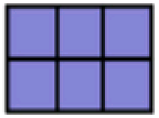


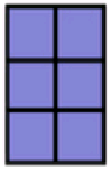
2 x 4

4 x 2

$2 \times 3 = 6 = 3 \times 2$

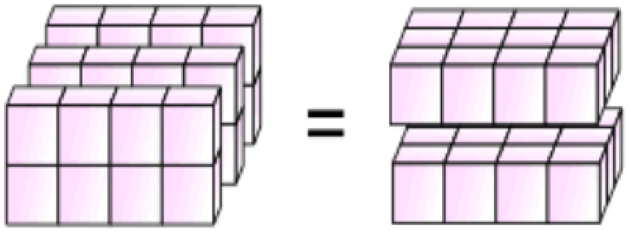


= 6



Multiplication is also **ASSOCIATIVE**.

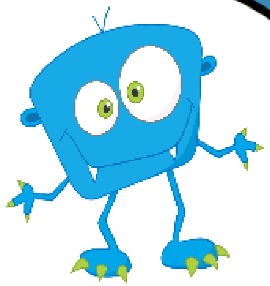
It doesn't matter how we group the numbers (i.e. which we calculate first) when we multiply.



$(2 \times 4) \times 3$ $\quad$ $2 \times (4 \times 3)$ $\quad$ **= 24**

The brackets show which calculation has been worked out first.

Here are three 1-digit cards.



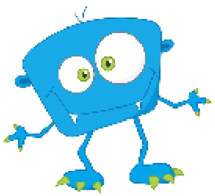
9	3	6

Arrange them to create a multiplication calculation and work out the answer.

☐ x ☐ x ☐ =

9 3 6

Now rearrange the cards to create 2 different calculations and work out the answer.

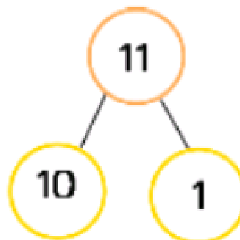


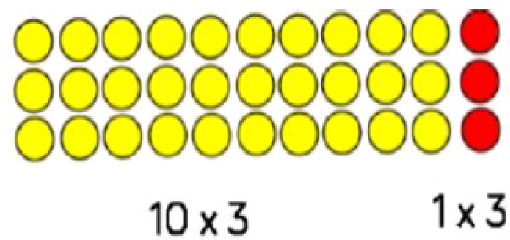
What do you notice about the three answers?

☐	x	☐	x	☐	=
☐	x	☐	x	☐	=
☐	x	☐	x	☐	=

11 x 3

11 x 3





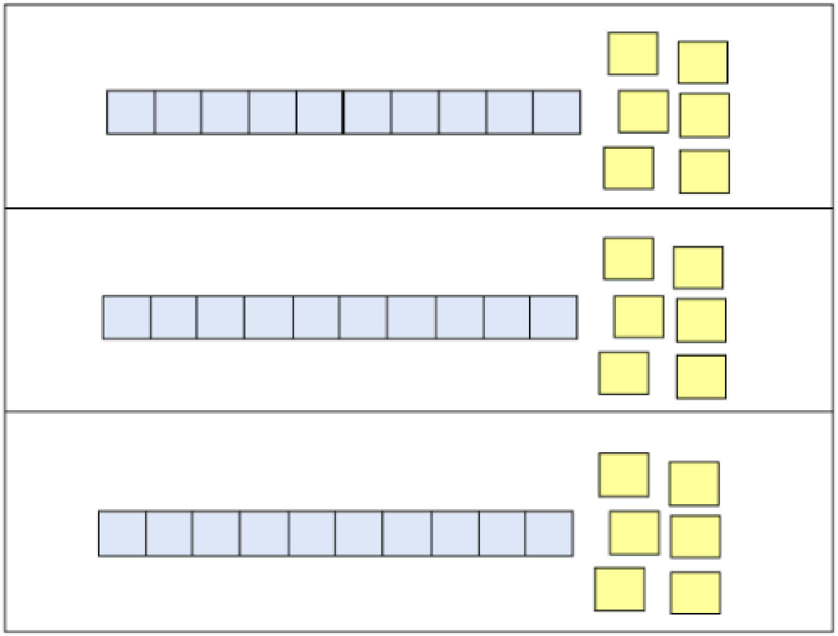
How has 11 been partitioned?

Name and explain each model.

Why has it been partitioned in this way?

Use partitioning to calculate this:

16 x 3



16 has been partitioned into ______ and ______.

Ten lots of 3 is ________

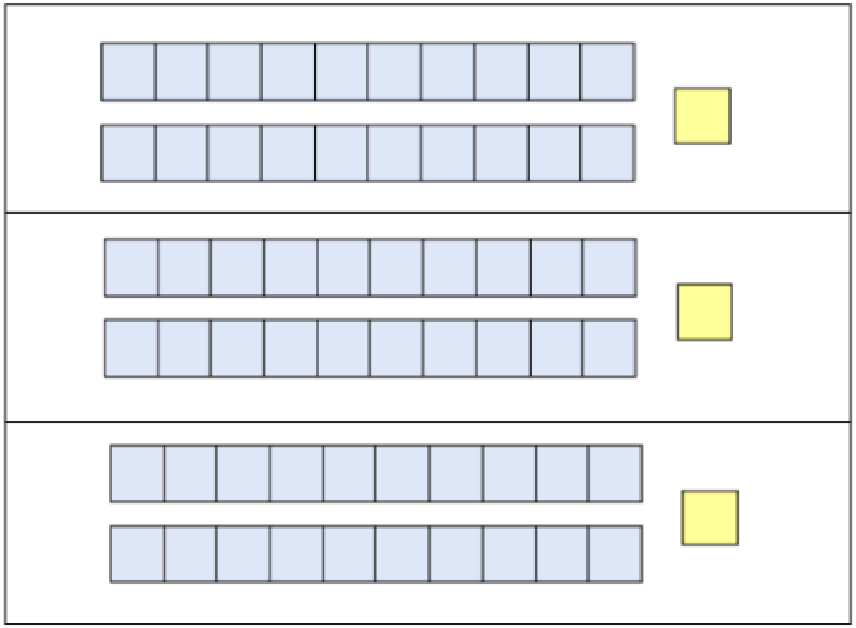
Six lots of 3 is ________

$10 \times 3 + 6 \times 3 =$ ________

$16 \times 3 =$ _______

Use partitioning to calculate this:

21 x 3



21 has been partitioned into ______ and ______.

Twenty lots of 3 is ________

One lot of 3 is ________

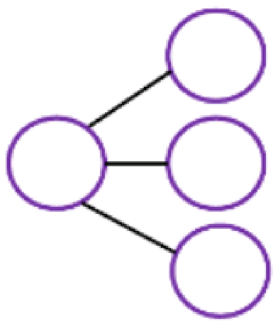
20 x 3 + 1 x 3 = ________

21 x 3 = ________

Multiplying using partitioning

Multiply. 123 x 3 = ☐

1. Complete the part-part whole diagram by partitioning 123 into hundreds, tens and ones. Then fill in the missing boxes.



☐ x 3 = ☐

☐ x 3 = ☐

☐ x 3 = ☐

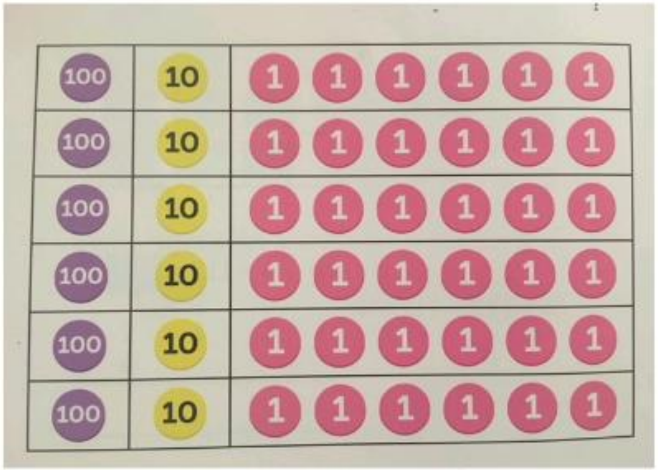
☐ + ☐ + ☐ = ☐

100	20	3
100	20	3
100	20	3

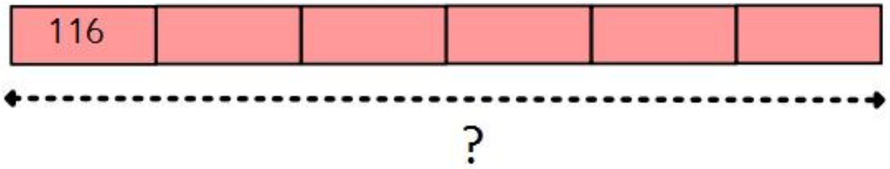
Problem solving

Each ticket from London to Middlesborough costs £116.

How can we find the cost of 6 tickets from London to Middlesborough?

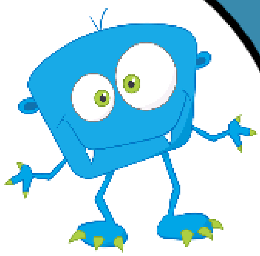


100 x 6 10 x 6 10 x 6



£116 x 6 =

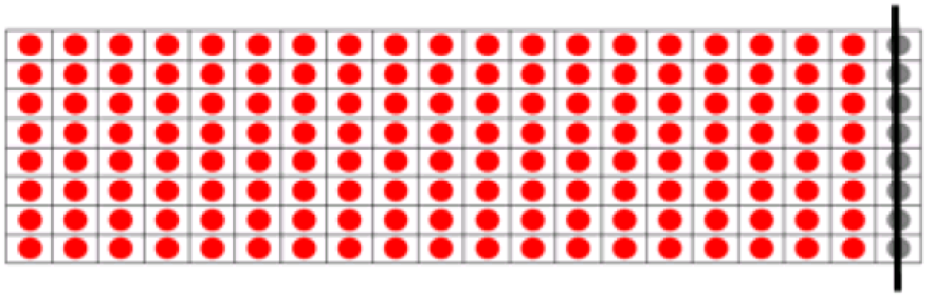
19 x 8



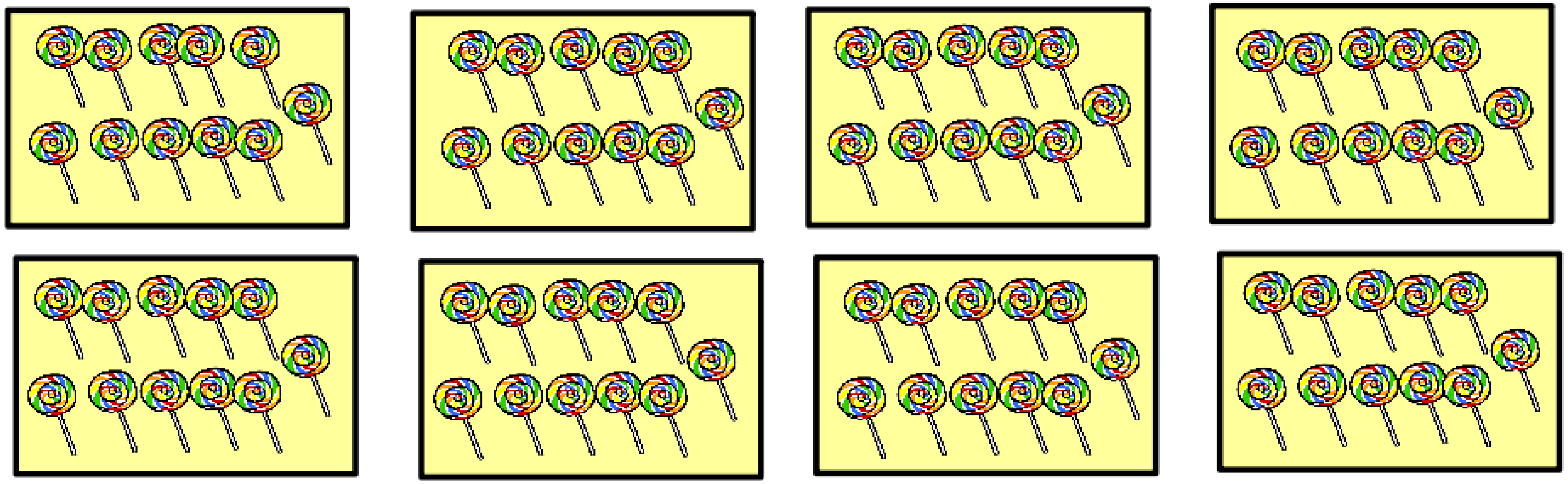
When a number is a near multiple, we mean that it is close to a multiple of 10.

19 is very close to 20. Multiplying by 20 would be much easier so we can do: **20 x 8** and then adjust.

$$19 \times 8 = 20 \times 8 - 1 \times 8$$

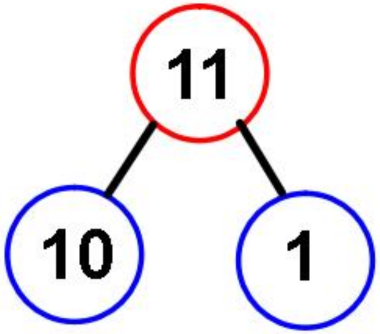


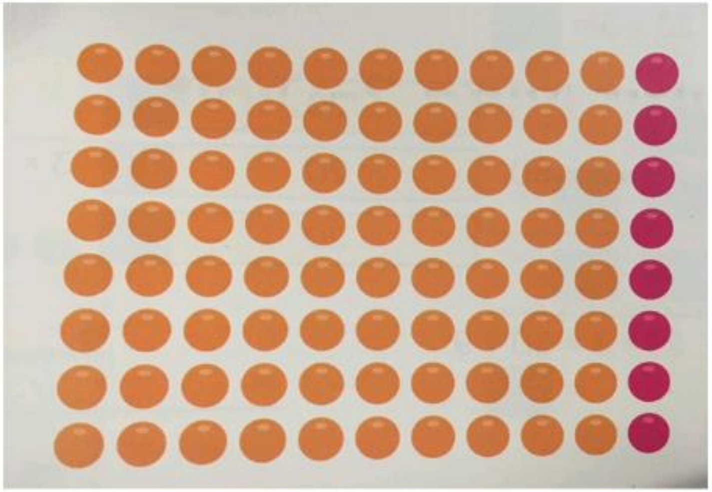
Multiplying 2 digit numbers.



8 x 11 = __________

Method 2:





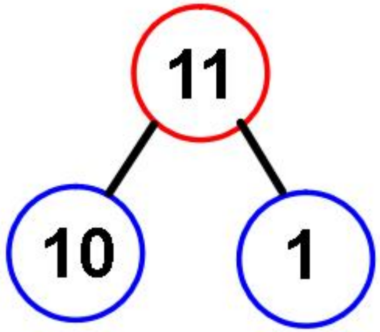
8 x 10 = 80

8 x 1 = 8

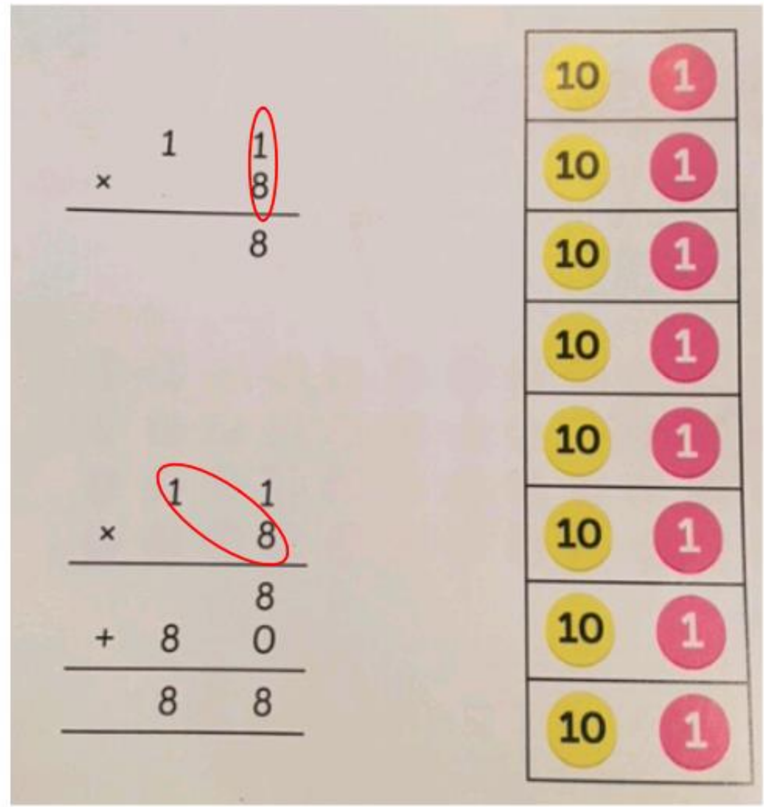
8 x 11 = 80 + 8

= ______

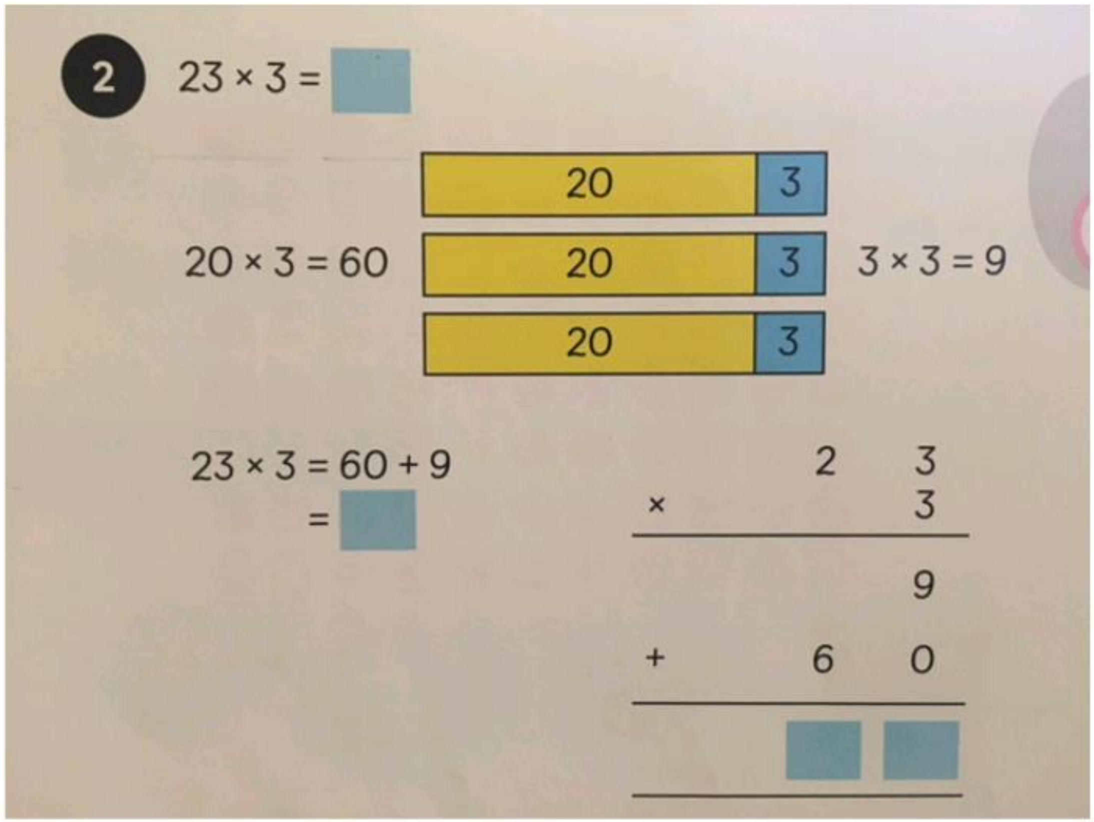
Method 3:



8 x 11 = 80
8 x 1 = 8



Let's practise using the written method.



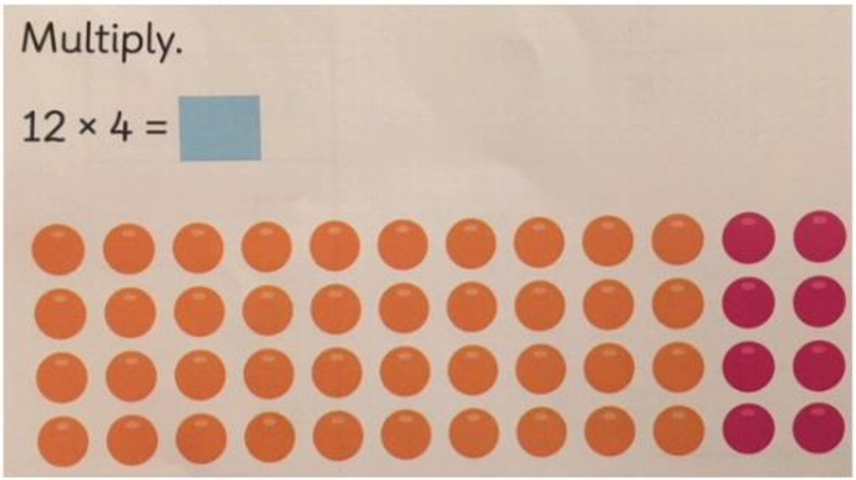
2 $23 \times 3 = \square$

	20	3
$20 \times 3 = 60$	20	3	$3 \times 3 = 9$
	20	3

$23 \times 3 = 60 + 9$

$= \square$

$$\begin{array}{rrr} & 2 & 3 \\ \times & & 3 \\ \hline & & 9 \\ + & 6 & 0 \\ \hline & \square & \square \\ \hline \end{array}$$



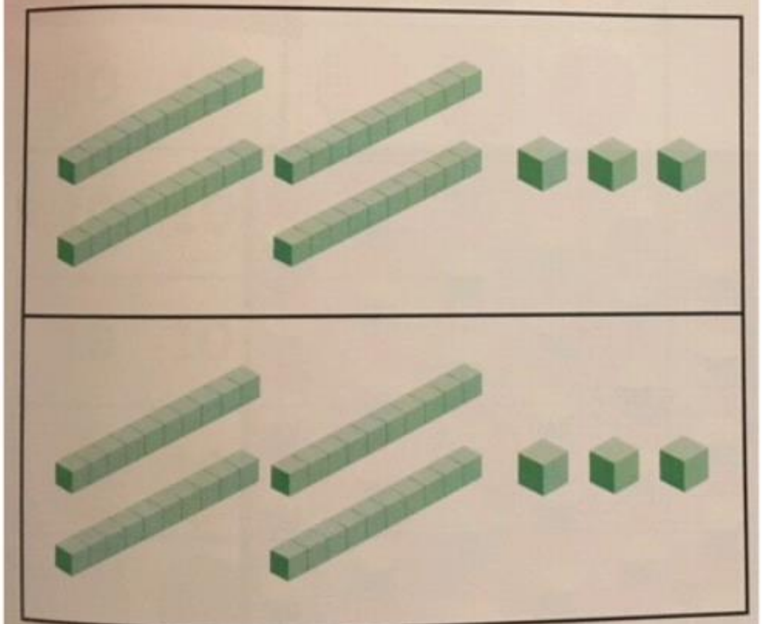
Multiply.

12 × 4 = $\boxed{}$

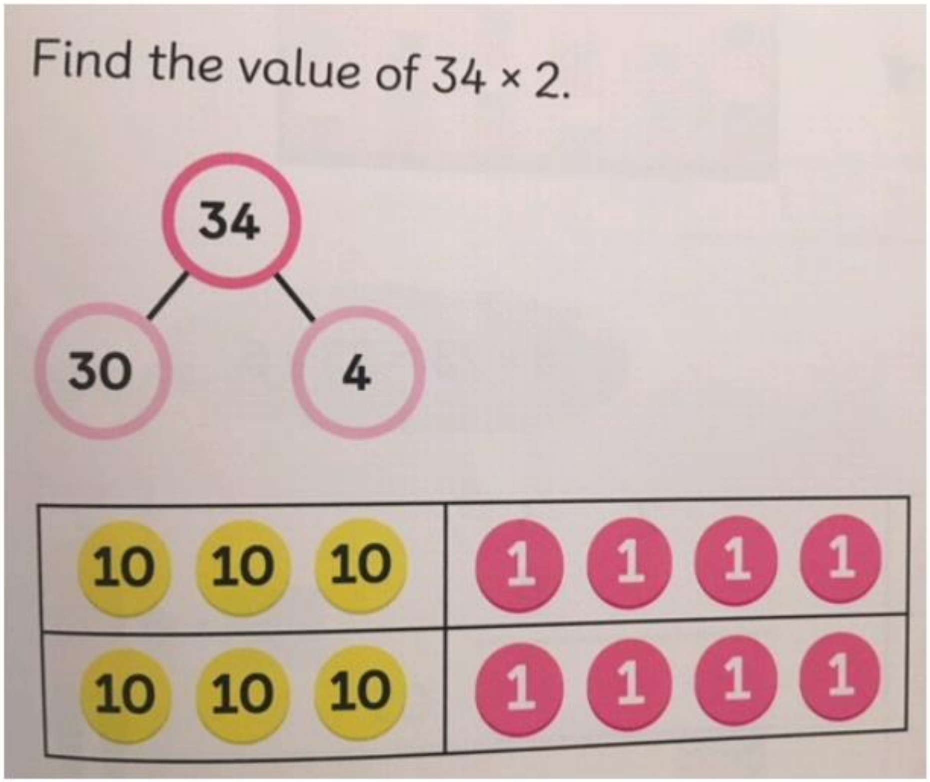
$$
\begin{array}{rrr}
 & 1 & 2 \\
\times & & 4 \\
\hline
 & & \square \\
+ & \square & \square \\
\hline
 & \square & \square
\end{array}
$$

Find the product of 43 and 2.

$43 \times 2 = \square$



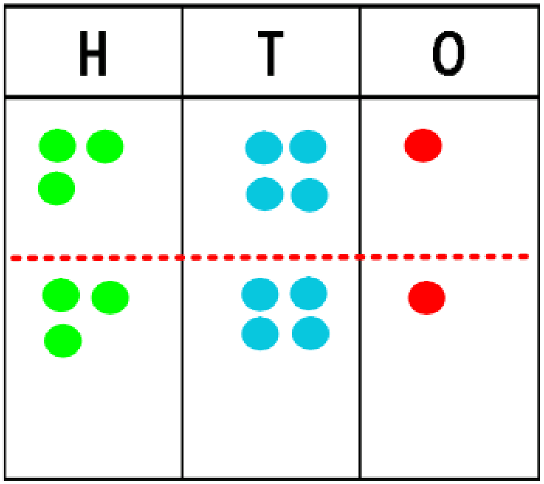
$$
\begin{array}{rcc}
 & 4 & 3 \\
\times & & 2 \\
\hline
 & & \square \\
+ & \square & \square \\
\hline
 & \square & \square
\end{array}
$$



$$\begin{array}{r} 3\ 4 \\ \times \quad\ 2 \\ \hline \end{array}$$

341 x 2

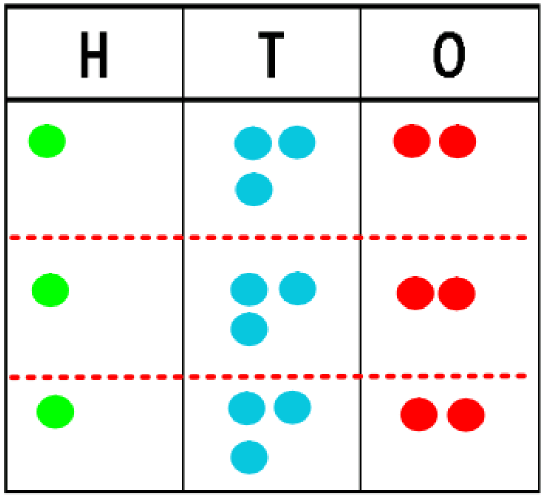
$$\begin{array}{r} 3\ 4\ 1 \\ \times \quad\quad 2 \\ \hline \\ \hline \end{array}$$



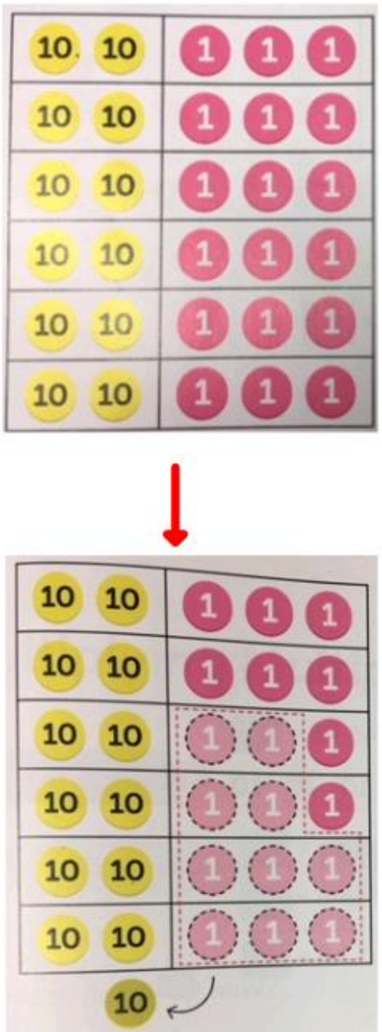
First multiply the ones.
Then multiply the tens.
Finally, multiply the hundreds.

132 x 3

$$\begin{array}{r} 1\ 3\ 2 \\ \times \quad\quad 3 \\ \hline \\ \hline \end{array}$$



First multiply the ones.
Then multiply the tens.
Finally, multiply the hundreds.



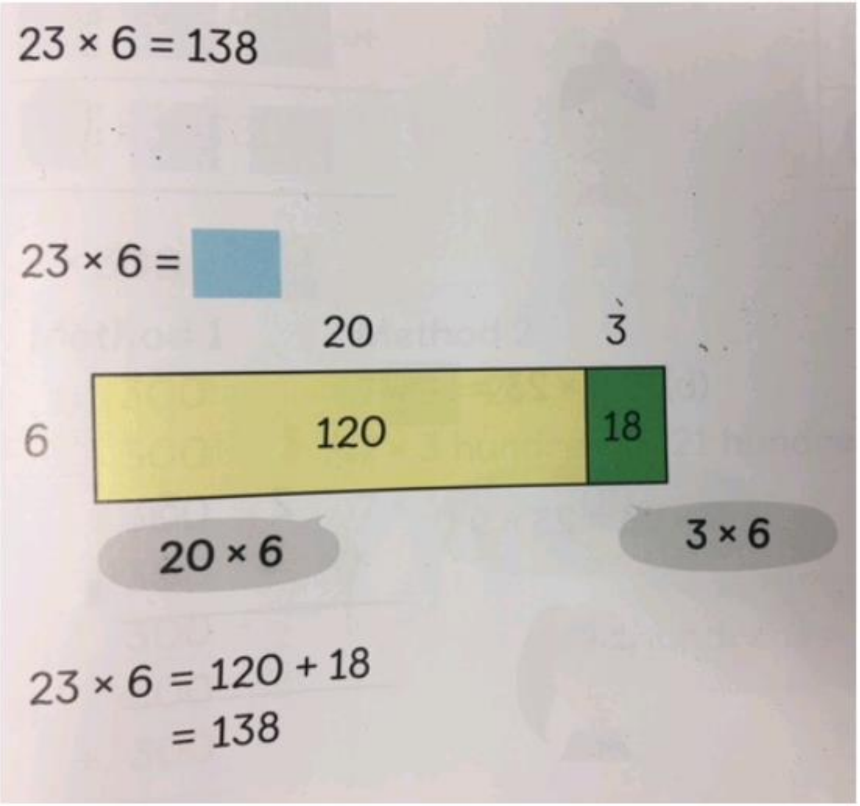
6 x 23 =

$$\begin{array}{r} 2\ 3 \\ \times\quad 6 \\ \hline \\ \hline \end{array}$$

6 x 23 = 23 x 6

Discuss what us happening to the numbers as you multiply each number.
Use the model to help you explain.

6 x 23 =

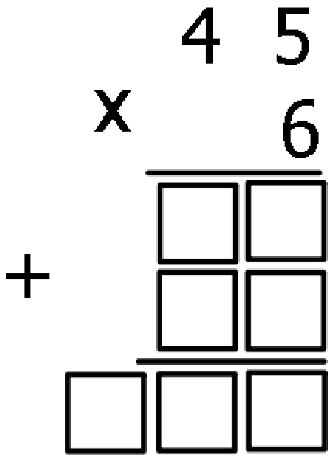


Multiply

45 x 6

As you use the strategy, explain what is happening the numbers as you multiply.

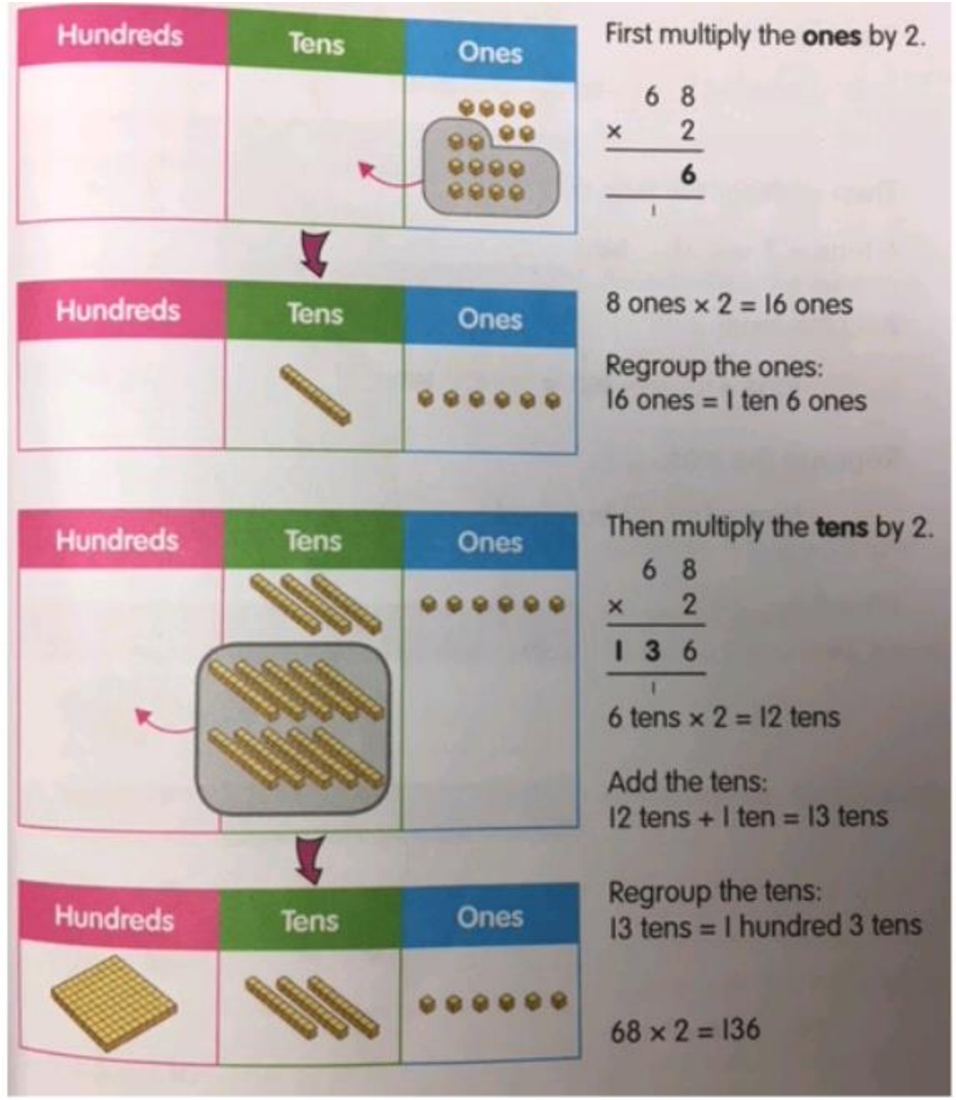
What numbers need exchanging? Why?

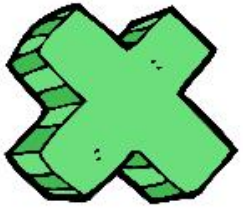


68 x 2

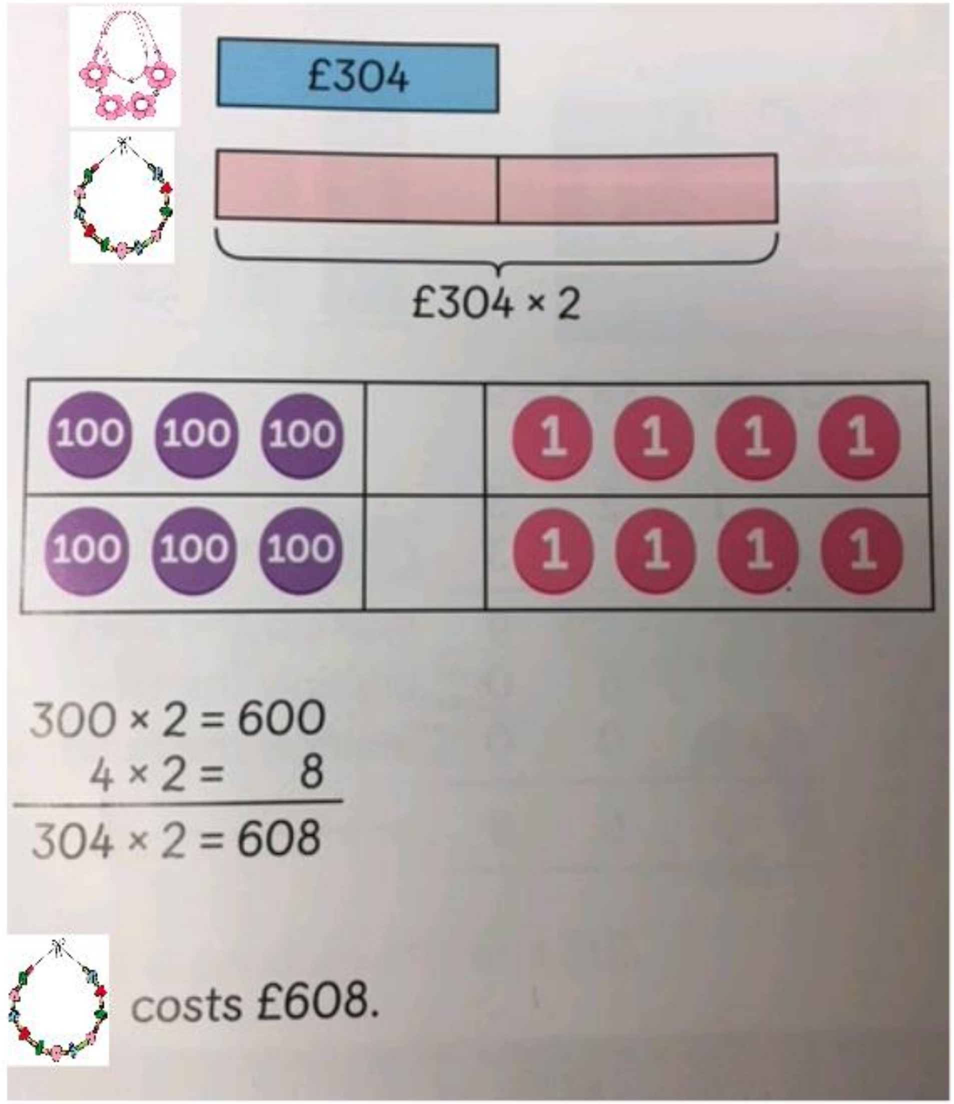
Let's discuss what happens at each stage of the multiplication.

The overall strategy:
multiply from right to left; regroup whenever it is necessary.





£304 x 2



£304

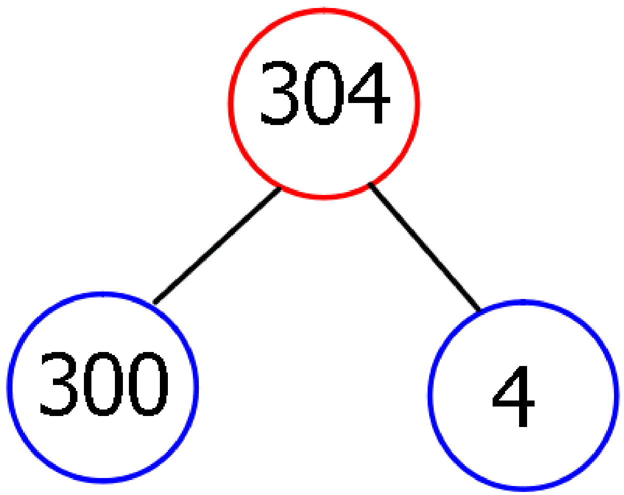
£304 × 2

100 100 100		1 1 1 1
100 100 100		1 1 1 1

$$
\begin{array}{r}
300 \times 2 = 600 \\
4 \times 2 = 8 \\
\hline
304 \times 2 = 608
\end{array}
$$

costs £608.

304 x 2 =

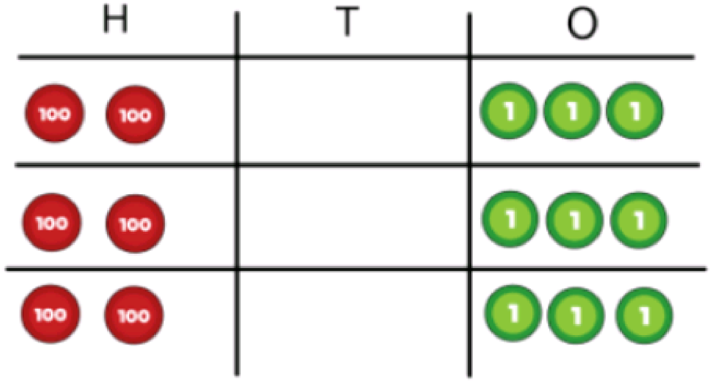


300 x 2 =

4 x 2 =

$$\begin{array}{rccc} & 3 & 0 & 4 \\ \times & & & 2 \\ \hline & & & \\ & & & \\ \hline & & & \end{array}$$

203 x 3 =



H	T	O
2	0	3
×		3

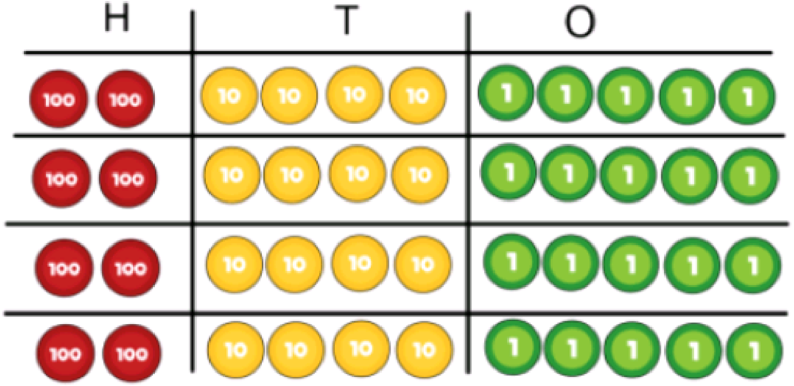
200 x 3 =

0 x 3 =

3 x 3 =

What happens when there is a 0 in the ones column, tens column or hundreds column?

A school has 245 packets of sweets.
Each packet contains 4 sweets.
How many sweets are there altogether?



H	T	O
2	4	5
×		4

Use the place value counters to solve the problem.
Remember, if there are ten or more counters in a column, to make an exchange.

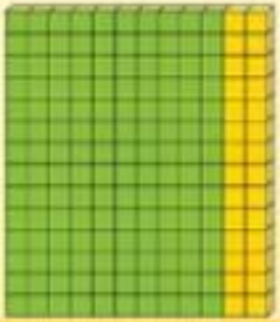
Multiplying a 2 digit number by a 2 digit number

Concrete materials

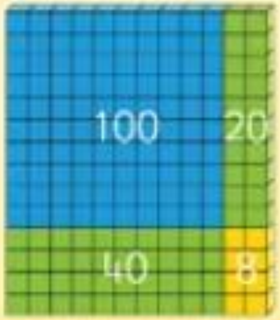
There are 12 eggs in this box. How many eggs in 14 boxes?

14 × 12 = ____

(a) We can use base ten blocks to make 14 rows of 12.



(b) We can use bigger blocks to show it more quickly. Here we have used one 100 block instead of ten 10 rods, and used two 10 rods instead of 20 unit cubes.

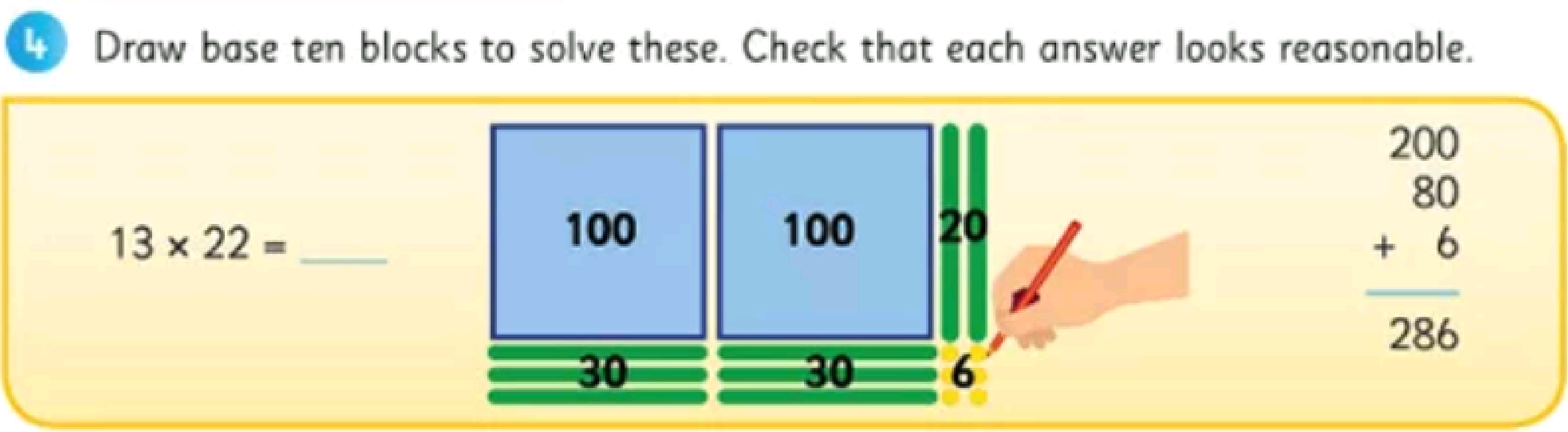


(c) We can total the value of the blocks to get the answer.

$$\begin{array}{r} 100 \\ 40 \\ 20 \\ +\ \ 8 \\ \hline 168 \end{array}$$

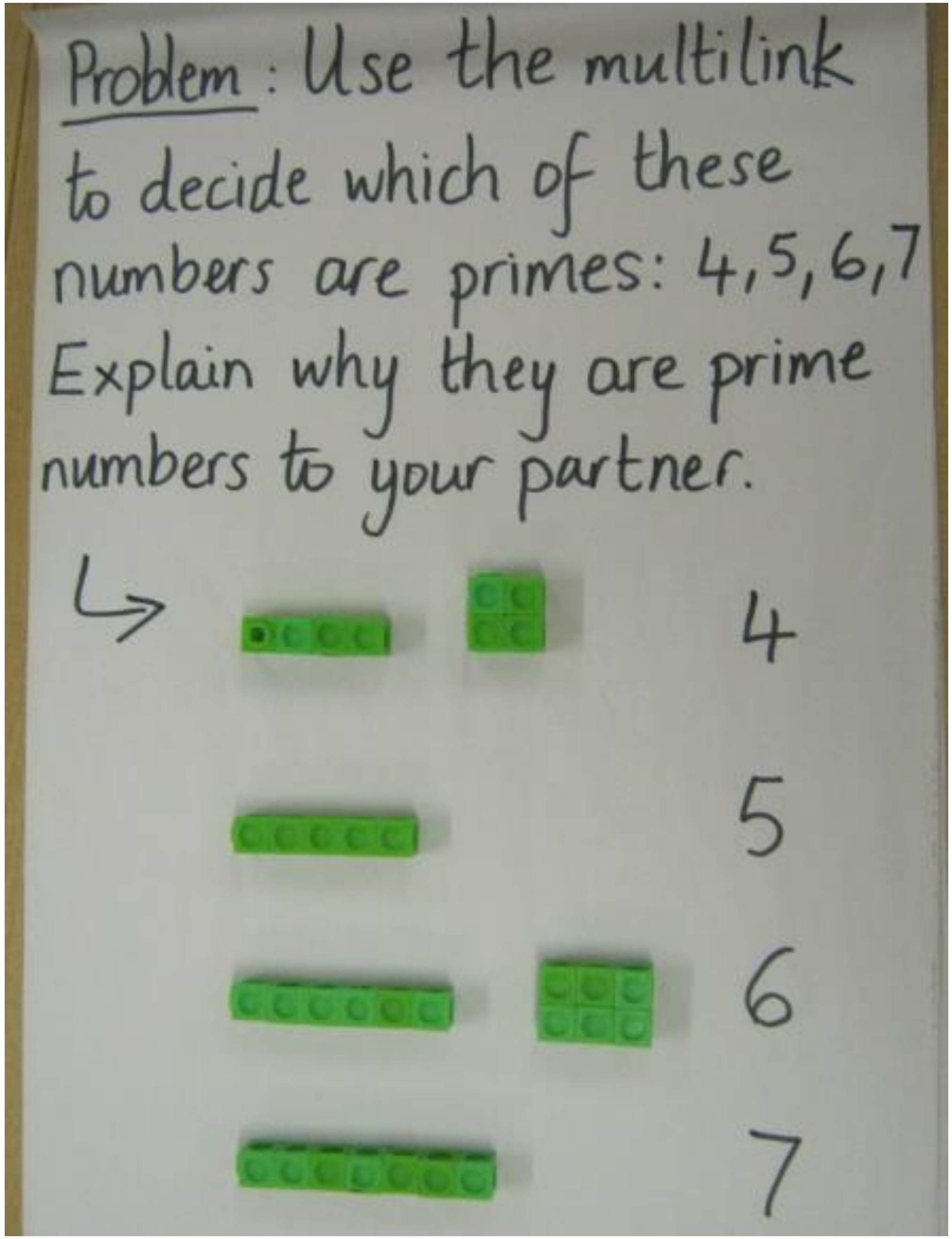
Using base ten blocks to demonstrate multiplication as an area array.

Pictorial

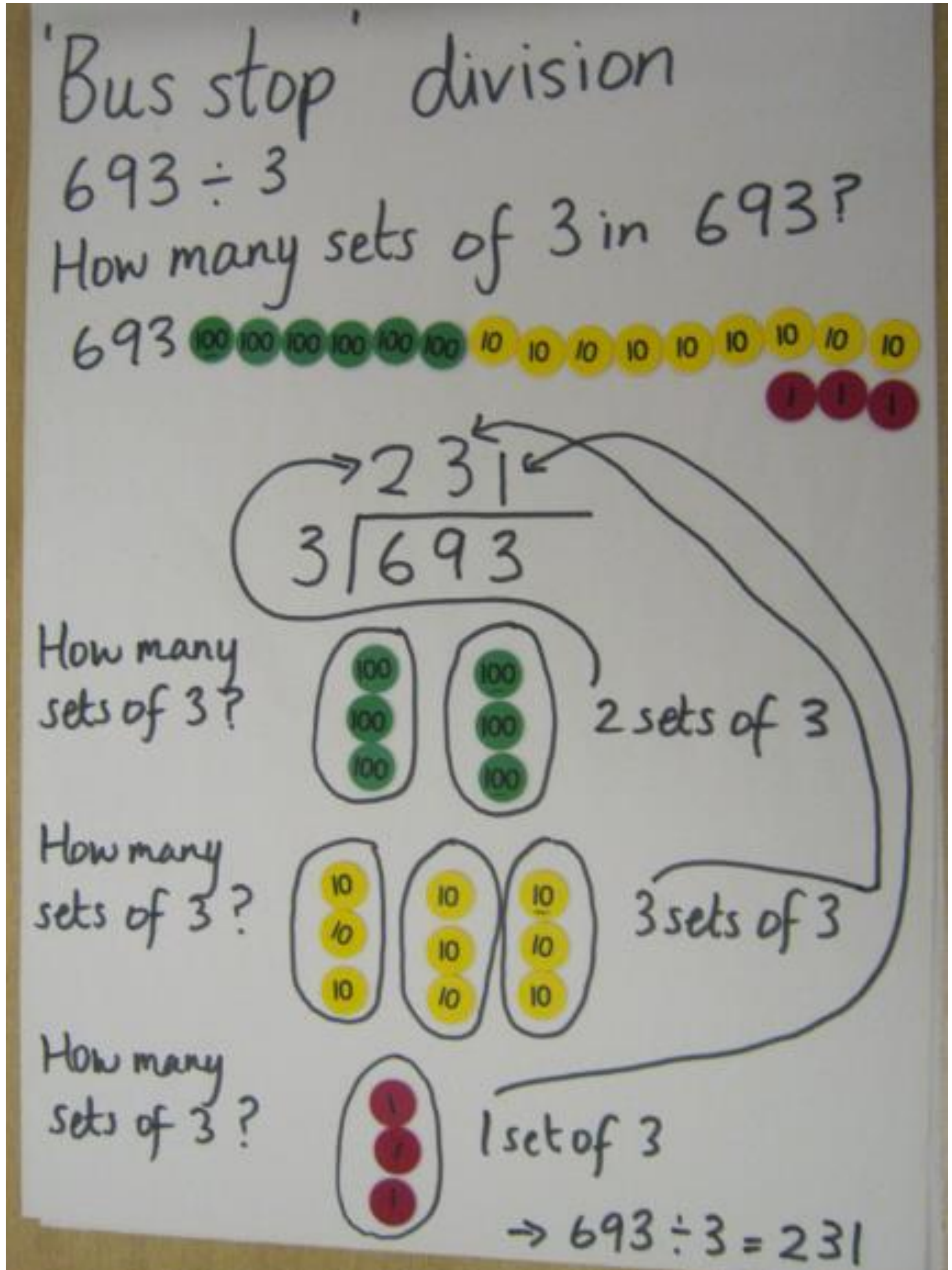


Moving on from the actual blocks; drawing a pictorial representation

CPA in KS2


Problem: Use the multilink to decide which of these numbers are primes: 4, 5, 6, 7
Explain why they are prime numbers to your partner.
4
5
6
7

CPA in Upper KS2



Let's explore some key mathematical concepts using the concrete materials (manipulatives)

Finding the difference

- A concept that begins in Nursery and is developed throughout each and every year group.
- This key concept is particularly difficult for children to grasp.
- To find the difference between 2 numbers, the operation we are using is subtraction.

Think about the manipulatives you have on your table, *how could you first demonstrate the idea of ‘difference’ to your child?*

What manipulatives could you use?

Do some make it clearer than others?

Key learning: to compare two sets using the language 'more', 'fewer' and 'difference'.

Develop Learning

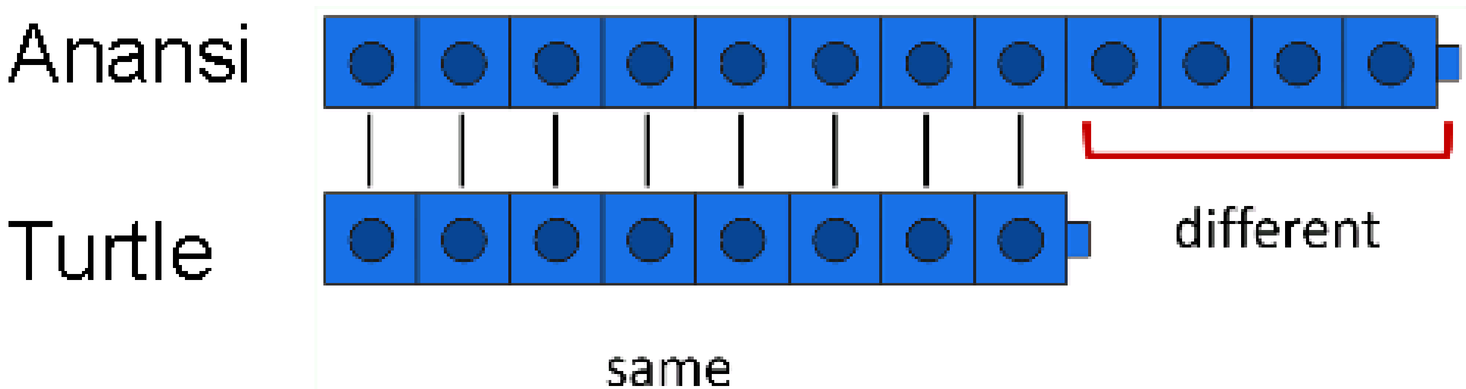
How many cakes are at Anansi's feast?
How many cakes are at Turtle's feast?

compare more fewer difference

Key learning: to compare two sets using the language 'more', 'fewer' and 'difference'.

Develop Learning

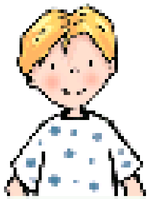
How many more cakes does Anansi have than Turtle?



- Anansi has 4 more cakes than Turtle.
- Turtle has 4 fewer cakes than Anansi.
- There is a difference of 4 between the number of cakes they have.

compare more fewer difference

Talk Task

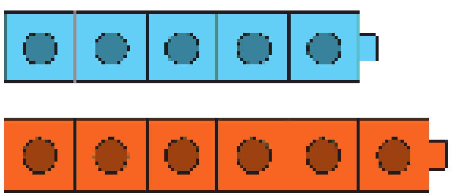


Adam is five years old. I will represent that using five cubes.

Anna is six years old. I will represent that using six cubes.

There is one cube sticking out. Anna is one year older than Adam. Adam is one year younger than Anna.

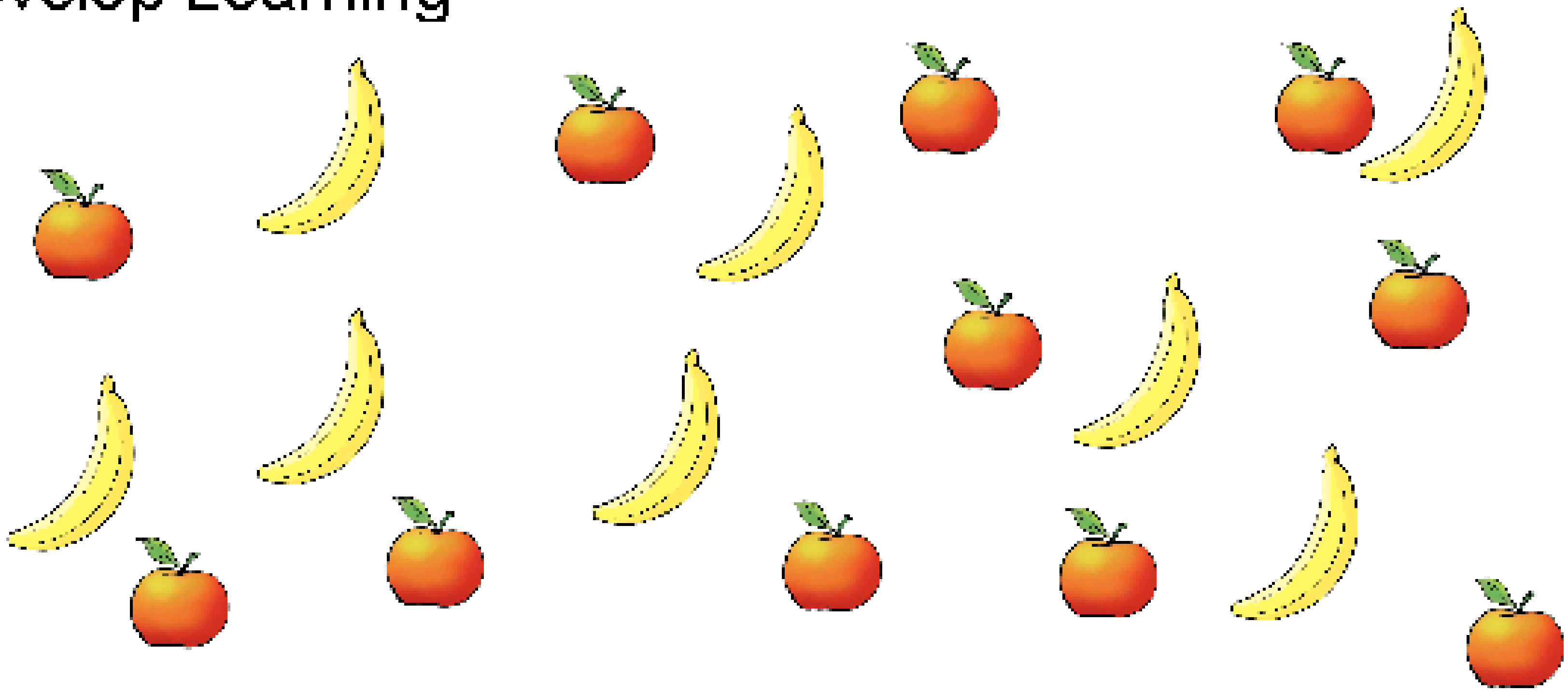
The difference between their ages is one year.



compare more fewer difference

Key learning: to compare two sets and find the difference in a range of contexts.

Develop Learning

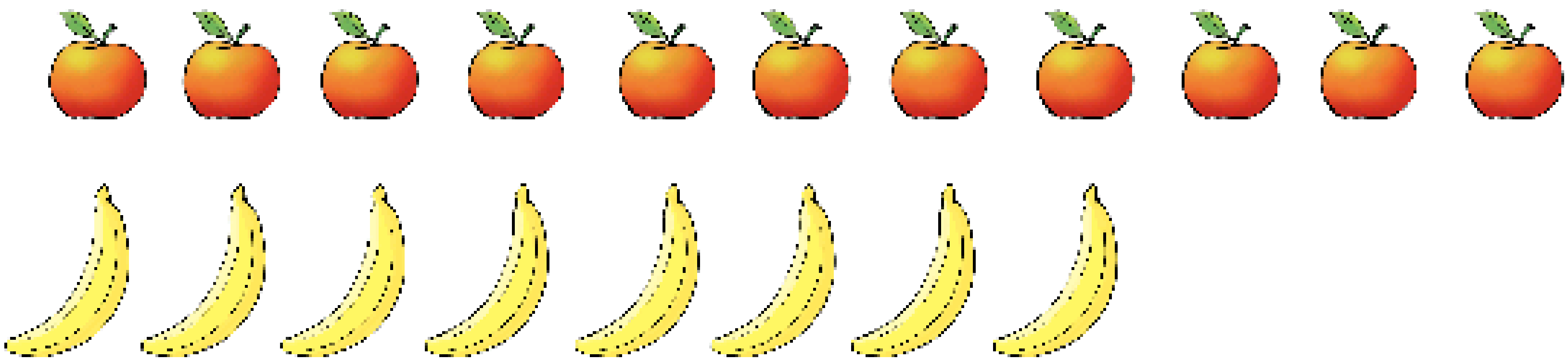


Are there more bananas or apples?

compare	more	fewer	difference

Key learning: to compare two sets and find the difference in a range of contexts.

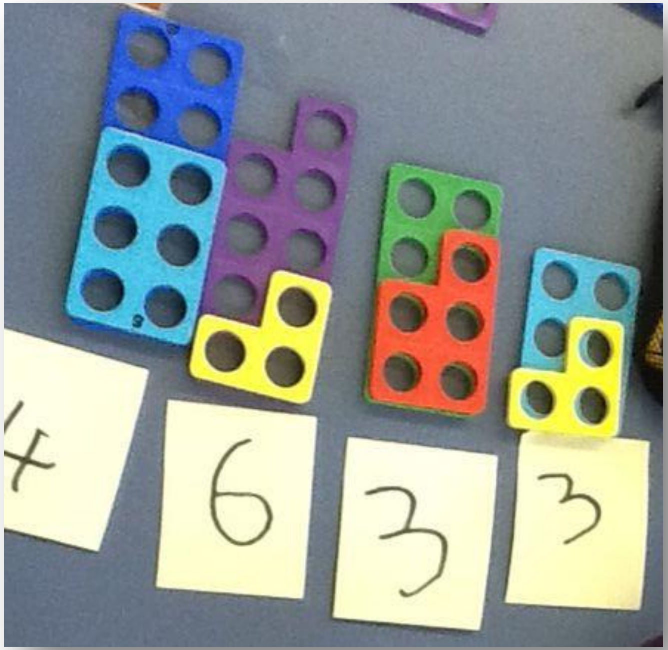
Develop Learning



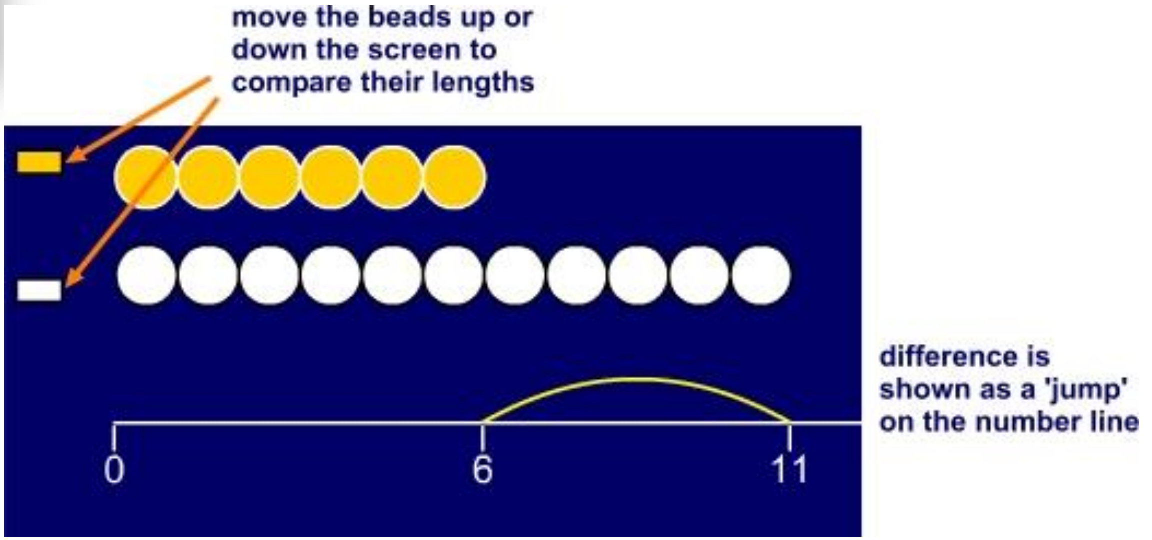
Are there more bananas or apples?

compare   more   fewer   difference

Finding the difference



Google: ITP maths for interactive teaching models.



Key learning: to compare two numbers using 'greater', 'less' and 'difference'.

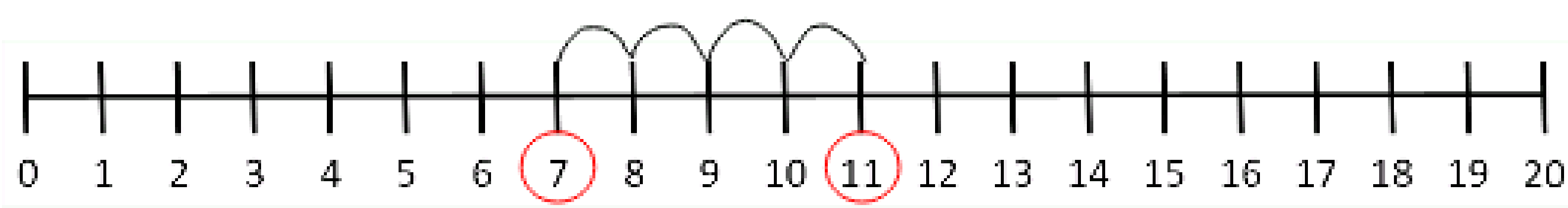
Talk Task

Number bonds to 6

I've rolled a 4. I'll start at 7. (Marking jumps) 1, 2, 3, 4.
4 greater than 7 is 11.

11 is 4 greater than 7. 7 is 4 less than 11.

The difference between 7 and 11 is 4. The difference between 11 and 7 is 4.



compare difference greater less

How would you use the bar model to represent finding the difference?

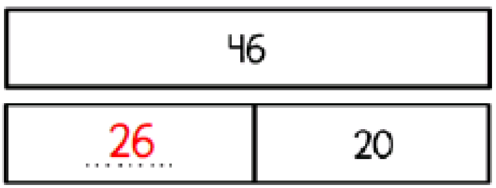
Bar Models 2

Name: ____________________

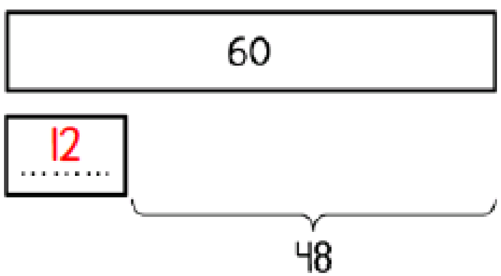
Teacher: ____________________

1 Work out the missing values.

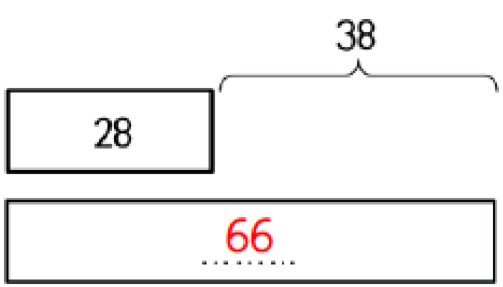
(a)



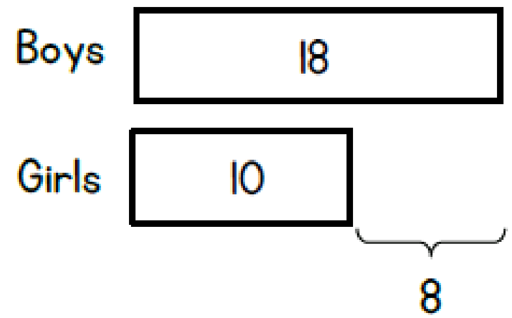
(b)



(c)



2 The bar model shows information about children in a class.



Complete the following sentences

There are ..18.... boys in the class.

There are ..10..... girls in the class.

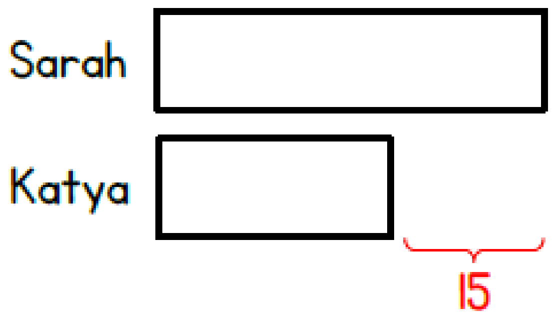
There are 8 ..more... boys in the class than girls.

There are 8 ..fewer.. girls in the class than boys.

There are ..28.... children in the class.

3 Sarah has 15 more flowers than Katya.

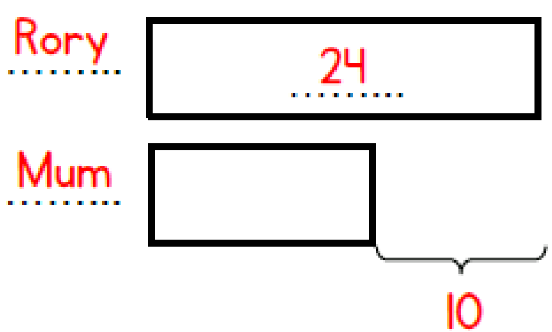
Show this on the diagram.



4 Rory bakes 24 buns.

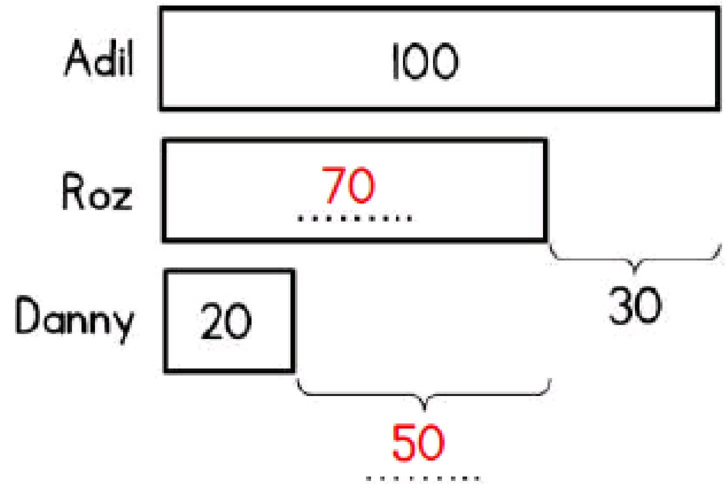
His Mum bakes 10 fewer buns.

Complete the diagram.



5 Adil, Roz and Danny have some bricks.

Complete the missing values.



Complete the following sentences

Adil has 100 bricks.

Roz has 30 ..fewer.. bricks than Adil.

Roz has ..50... more bricks than Danny.

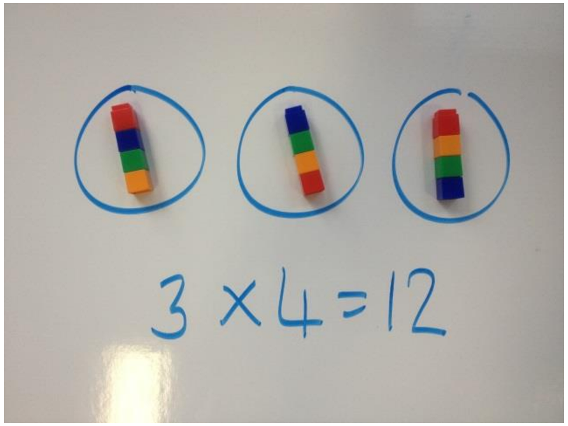
Adil has ..80 more.. bricks than Danny

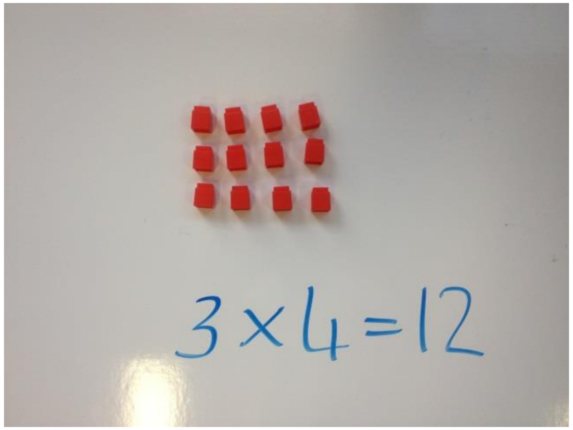
Activity 3

Look at a different statement below and use concrete manipulatives to model it.
How many different manipulatives can you use and what learning occurs, or what is being reinforced?

- 3 x 4 = 12
- Compare 31 and 35
- Subtraction: 40 – 7 and 43 - 28
- 13 x 22 = 286

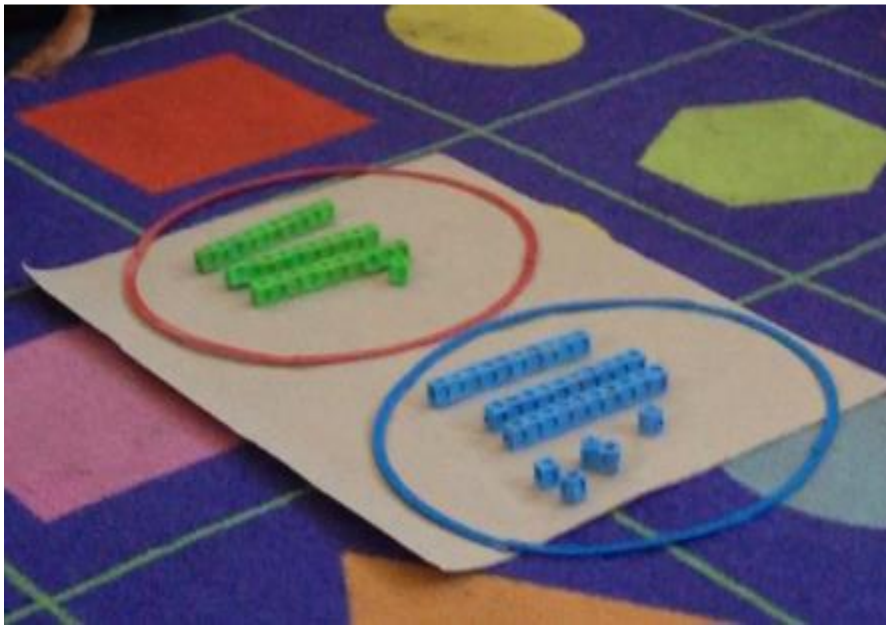
3 x 4 = 12





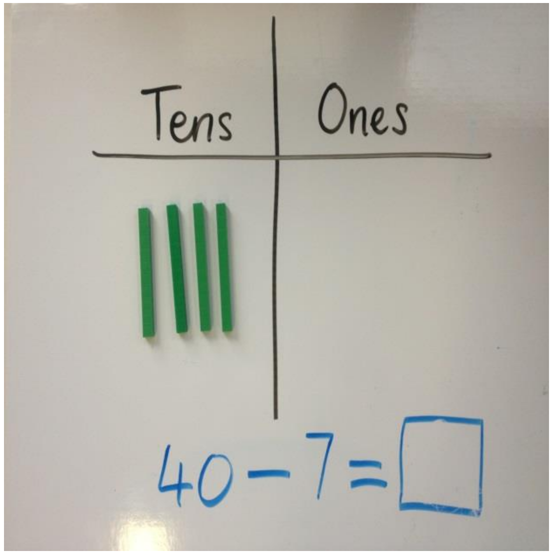
- Pupils can count the total
- Reinforces ‘groups of’
- It is clear that 4x4 would require another group/row of 4 (repeated addition)

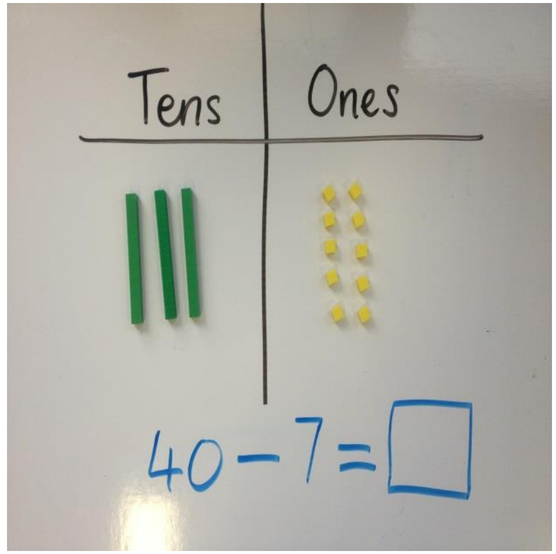
Compare 31 and 35

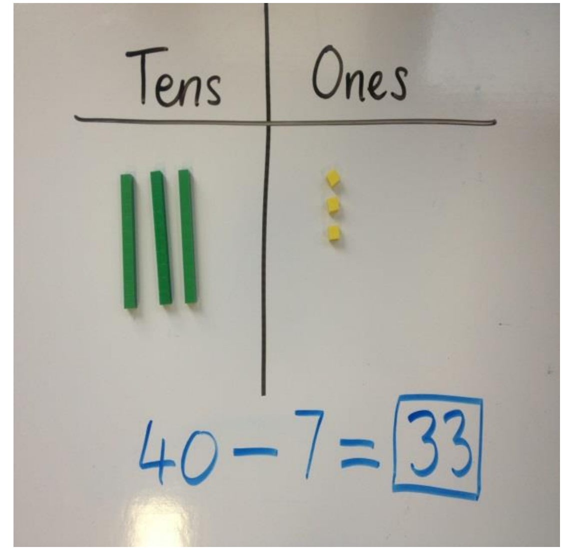


- Comparing numbers
- More or fewer
- Look at the tens and ones
- What's the same and what's different?

40 – 7 = 33

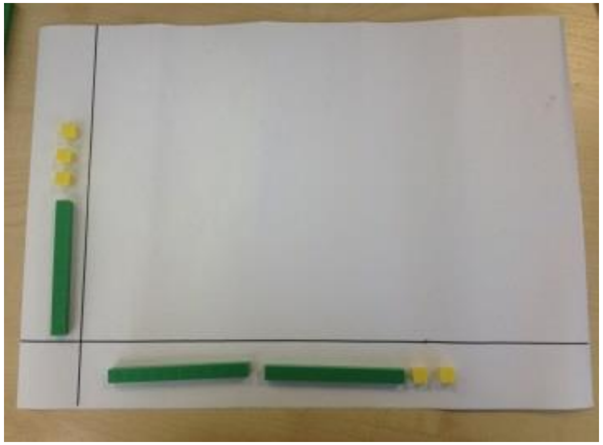


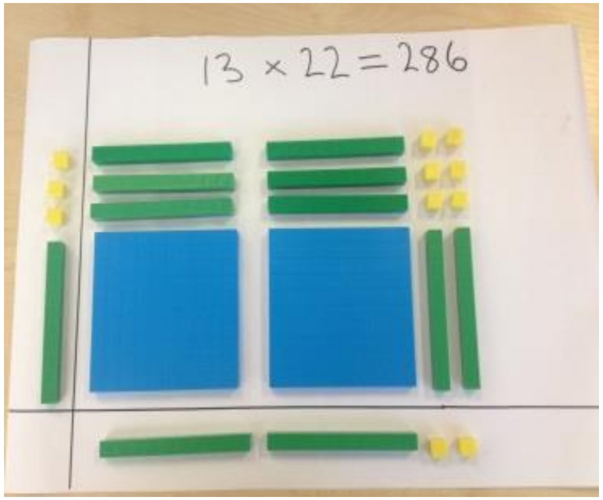




- Understanding re-grouping
- Reinforcing place value

13 x 22 = 286





- Supports counting in 100s, 10s and 1s
- Shows that every number is multiplied by every other number

Activity 4

Discuss:

- Why do you think the use of concrete manipulatives reduces as pupils journey through education?
- Why is this concerning?

C-P-A benefits

- Provides pupils with a structured way to learn maths concepts
- Pupils can build a better connection when moving through the levels of understanding from concrete to abstract
- Makes learning accessible to all learners
- Research has proven this method is effective
- Able to use across year groups, from early primary through secondary school
- Helps pupils learn concepts before learning rules
- Can be used in small groups or the entire class
- Can be used to differentiate and to assess true understanding – MASTERY!